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EXTERIOR AXISYMMETRIC MIXED PROBLEM FOR A TRANSVERSELY ISOTROPIC HALF-SPACE

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We obtain in closed form the exact solution of the axisymmetric mixed problem for a transversely isotropic half-space, when the normal and the shear stresses are given inside a circle, while in the exterior of the circle the normal and radial displacements are prescribed.

Several papers [1 – 5] have been devoted to the mixed problems of the theory of elasticity for an isotropic half-space. The related problems for the transversely isotropic half-space have been studied to a lesser extent [6, 7].

1. We consider a transversely isotropic elastic half-space $z \geqslant 0$, whose isotropy planes are parallel to the boundary. We assume that the normal stress $\sigma_0(\rho)$ and the radial shear stress $\tau_0(\rho)$ are prescribed inside a circle of radius a; the normal displacement $\varphi_1(\rho)$ and the radial displacement $\varphi_2(\rho)$ are prescribed outside the circle. We consider the problem of the determination of the stresses $\sigma(\rho)$ and $\tau(\rho)$ for $\rho > a$. By methods similar to those applied in [8], we can reduce the problem under consideration to a system of two integral equations relative to the desired stresses

$$2H\left[-\pi\alpha\int_\rho^\infty \tau(x)\,dx + 2\int_\rho^\infty \frac{du}{\sqrt{u^2-\rho^2}}\int_a^u \frac{\sigma(x)\,x\,dx}{\sqrt{u^2-x^2}}\right] = \psi_1(\rho) \tag{1.1}$$

$$2H\left[2\gamma_1\gamma_2\rho\int_\rho^\infty \frac{du}{u^2\sqrt{u^2-\rho^2}}\int_a^u\frac{\tau(x)\,x^2dx}{\sqrt{u^2-x^2}} - \frac{\pi\alpha}{\rho}\int_a^\rho \sigma(x)\,x\,dx\right]=\psi_2(\rho)$$

$$\psi_1(\rho)=\varphi_1(\rho)-4H\int_0^a\frac{du}{\sqrt{\rho^2-u^2}}\int_u^a\frac{\sigma_0(x)\,x\,dx}{\sqrt{x^2-u^2}}$$

$$\psi_2(\rho)=\varphi_2(\rho)-4\gamma_1\gamma_2\frac{H}{\rho}\int_0^a\frac{u^2du}{\sqrt{\rho^2-u^2}}\int_u^a\frac{\tau_0(x)\,dx}{\sqrt{x^2-u^2}}+2\pi\frac{H\alpha}{\rho}\int_0^a\sigma_0(x)\,x\,dx$$

The constants H, α, γ_1, γ_2 are determined in terms of the elastic constants of the material of the half-space [8].

We will seek the solution of the system (1.1) in the form

$$\sigma(x) = \frac{1}{x}\frac{d}{dx}\left[\int_x^\infty \frac{f_1(t)\,t\,dt}{\sqrt{t^2-x^2}} + \int_a^x \frac{f_2(t)\,t\,dt}{\sqrt{x^2-t^2}}\right] \tag{1.2}$$

$$\tau(x) = \frac{d}{dx}\left[C_1\int_a^x \frac{f_1(t)\,dt}{\sqrt{x^2-t^2}} + C_2\int_x^\infty \frac{f_2(t)\,dt}{\sqrt{t^2-x^2}}\right] + \frac{C_1 D a}{x\sqrt{x^2-a^2}}$$

Here $f_{1,2}(t)$ are new unknown functions; the constants C_1, C_2, D will be determined in the sequel. The substitution of the solutions (1.2) into the system (1.1), after interchanging the order of integration and computing the obtained inner integrals, leads us to two equations relative to $f_1(t)$ and $f_2(t)$, which can be solved independently if we set $C_1 = \alpha / \gamma_1 \gamma_2$ and $C_2 = -1/\alpha$

$$2H\left[-\pi \frac{\alpha^2}{\gamma_1\gamma_2}\left(D \arcsin \frac{a}{\rho} - \int_a^{\rho} \frac{f_1(t)\,dt}{\sqrt{\rho^2 - t^2}} \right) + 2\int_{\rho}^{\infty} \frac{\sqrt{u^2-a^2}}{\sqrt{u^2-\rho^2}}\,du \int_a^{\infty} \frac{f_1(t)\,t\,dt}{\sqrt{t^2-a^2}\,(t^2-u^2)} \right] = \psi_1(\rho)$$

$$2\pi \frac{H\alpha}{\rho}\left[\int_a^{\infty}\left(\frac{t}{\sqrt{t^2-a^2}} - 1\right) f_1(t)\,dt + aD - \int_a^{\rho} \frac{f_2(t)\,t\,dt}{\sqrt{\rho^2-t^2}} - \right. \tag{1.3}$$

$$\left. \frac{2}{\pi}\frac{\gamma_1\gamma_2}{\alpha^2}\int_{\rho}^{\infty} \frac{du}{\sqrt{u^2-\rho^2}\,\sqrt{u^2-a^2}} \int_a^{\infty} \frac{u^4(t^2-a^2) - \rho^2 a^2 (t^2-u^2)}{u^2\sqrt{t^2-a^2}\,(t^2-u^2)} f_2(t)\,dt \right] = \psi_2(\rho)$$

2. Let us find the solution of the first equation of (1.3). We divide both of its sides by $\rho\,(\rho^2 - r^2)^{1/2}$, integrate with respect to ρ from r to ∞, multiply the result by r and finally differentiate with respect to r. After some transformations we obtain the equation

$$\lambda^2 \int_a^{\infty} \Phi_1(r,t)\,dt - \int_a^{\infty} F_1(r,t)\,dt = \chi_1(r) \tag{2.1}$$

$$\chi_1(r) = \frac{1}{2\pi H}\frac{d}{dr}\left(r\int_r^{\infty} \frac{\psi_1(\rho)\,d\rho}{\rho\sqrt{\rho^2 - r^2}} \right) - \lambda^2 \frac{D}{r} \ln \sqrt{\frac{r+a}{r-a}}$$

$$\Phi_1(r,t) = \frac{f_1(t)}{t^2 - r^2}, \qquad \lambda^2 = \frac{\alpha^2}{\gamma_1\gamma_2}$$

$$F_1(r,t) = \frac{t\sqrt{r^2-a^2}}{r\sqrt{t^2-a^2}}\,\Phi_1(r,t)$$

We introduce the notation

$$Y_{c,s}(r) = \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right. \left[\theta \ln \frac{r+a}{r-a} \right]$$

The constant θ will be determined in the sequel. We multiply both sides of Eq. (2.1) by $Y_c(r) / (r^2 - x^2)$ and integrate with respect to r from a to ∞, making use of the Poincaré-Bertrand permutation formula [10]. After some transformations we obtain

$$\frac{\pi^2(1-\lambda^2)}{4x^2} Y_c(x) f_1(x) + \frac{\pi}{2}\lambda^2 \operatorname{cth} \pi\theta \int_a^{\infty} \Phi_1(x,t)\left[\frac{Y_s(x)}{x} - \frac{Y_s(t)}{t} \right] dt -$$

$$\frac{\pi a}{2x^2 \operatorname{ch} \pi\theta} \int_a^{\infty} \frac{f_1(t)\,dt}{t\sqrt{t^2-a^2}} - \operatorname{th}\pi\theta \int_a^{\infty} \left[F_1(x,t)\frac{Y_s(x)}{x} - \right. \tag{2.2}$$

$$\left. \Phi_1(x,t)\frac{Y_s(t)}{t} \right] dt = \frac{1}{x} X_1^c(x)$$

Here and in the following we have denoted

$$X_{1,2}^{c,s}(x) = \int_a^{\infty} \frac{\chi_{1,2}(r)}{r^2 - x^2} \left\{ \begin{matrix} x\,Y_c(r) \\ r\,Y_s(r) \end{matrix} \right. dr$$

We select θ so that we should have

$$\lambda^2 \operatorname{cth} \pi\theta - \operatorname{th} \pi\theta = 0, \qquad \theta = \pi^{-1} \operatorname{Arth} \lambda$$

Then the expression (2.2) is simplified to

$$\frac{\pi^2}{4x^2 \operatorname{ch}^2 \pi\theta} Y_c(x) f_1(x) + \frac{\pi}{2} \operatorname{th} \pi\theta \frac{Y_s(x)}{x} \int_a^\infty [\Phi_1(x, t) - F_1(x, t)]\, dt -$$

$$\frac{\pi a}{2x^2 \operatorname{ch} \pi\theta} \int_a^\infty \frac{f_1(t)\, dt}{t \sqrt{t^2 - a^2}} = \frac{1}{x} X_1^c(x) \tag{2.3}$$

Multiplying both sides of Eq. (2.1) by $r Y_s(r) / (r^2 - x^2)$ and performing the computations similar to those indicared above, we obtain

$$\frac{\pi^2}{4x \operatorname{ch}^2 \pi\theta} Y_s(x) f_1(x) - \frac{\pi}{2} \operatorname{th} \pi\theta\, Y_c(x) \int_a^\infty [\Phi_1(x, t) - F_1(x, t)]\, dt = X_1^s(x) \tag{2.4}$$

Now Eqs. (2.3) and (2.4) yield

$$f_1(x) = \frac{4 \operatorname{ch}^2 \pi\theta}{\pi^2} x [X_1^c(x) Y_c(x) + X_1^s(x) Y_s(x)] \tag{2.5}$$

3. By the same method, the second of the equations (1.3) reduces to the form

$$-\int_a^\infty \frac{f_2(t)\, t dt}{t^2 - r^2} + \frac{1}{\lambda^2}\left[r \sqrt{r^2 - a^2} \int_a^\infty \frac{f_2(t)\, dt}{\sqrt{t^2 - a^2}\,(t^2 - r^2)} + \right. \tag{3.1}$$

$$\left. + \int_a^\infty \frac{f_2(z)\, dt}{\sqrt{t^2 - a^2}} \right] = \frac{1}{2\pi H \alpha} \frac{d}{dr} \left(r \int_r^\infty \frac{\psi_2(\rho)\, d\rho}{\sqrt{\rho^2 - r^2}} \right) = \chi_2(r)$$

The exact solution of Eq. (3.1) is given by the formula

$$f_2(x) = -\frac{4 \operatorname{sh}^2 \pi\theta}{\pi^2} [X_2^c(x) Y_c(x) + X_2^s(x) Y_s(x)] \tag{3.2}$$

It should be noted that the solution (3.2) satisfies the second of the equations (1.3) only within terms of the form const / ρ, which have been lost in the process of solving by differentiation. However, we can always select the quantity D so that the equation be satisfied identically. The obtained solution can be used also in the case of complete isotropy if we set

$$H = \frac{1 - \nu^2}{\pi E}, \qquad \alpha = \frac{1 - 2\nu}{2(1 - \nu)}, \qquad \gamma_1 = \gamma_2 = 1$$

4. We consider an example: let us assume that a uniformly distributed normal load $\sigma_0 = q$ acts on the surface of the circle $\rho \leqslant a$ and that the shear stresses are vanishing; the remaining part of the boundary of the half-space is rigidly fixed. We determine the shear and normal stresses for $\rho > a$ and also the displacements inside the circle. Making use of the formulas obtained earlier, we have

$$\psi_1(\rho) = -4qH \int_0^a \frac{\sqrt{a^2 - u^2}}{\sqrt{\rho^2 - u^2}}\, du, \qquad \psi_2(\rho) = \pi q H \alpha \frac{a^2}{\rho}$$

$$\chi_1(r) = 2\pi q H \left(1 - \frac{\sqrt{r^2 - a^2}}{r}\right) - \lambda^2 \frac{D}{r} \ln \sqrt{\frac{r + a}{r - a}} \tag{4.1}$$

$$f_1(t) = \frac{2}{\pi} q \operatorname{cth} \pi\theta\, [t Y_s(t) - 2a\theta Y_c(t)] - D[1 - Y_c(t)] \qquad \chi_2(r) = f_2(t) = 0$$

The substitution of the expressions (4.1) into the second equation of (1.3) allows us to determine

$$D = \frac{2}{\pi} qa\theta \operatorname{cth} \pi\theta \tag{4.2}$$

The formulas (1.2), (4.1), (4.2) determine completely the stresses in the fixed part of the boundary of the half-space. Now we can determine the displacements inside the circle $\rho \leqslant a$ in the form

$$u = 2\pi qH\alpha \left\{ \frac{2}{\pi} \frac{\operatorname{cth} \pi\theta}{\rho} \left[\int_a^\infty \frac{tY_s(t) - a\theta[1 + Y_c(t)]}{\sqrt{t^2 - \rho^2}} t dt - a\theta \sqrt{a^2 - \rho^2} \right] - \frac{\rho}{2} \right\}$$

$$w = 4qH \left[\int_a^\infty \frac{\sqrt{t^2 - a^2} - tY_c(t) - a\theta Y_s(t)}{\sqrt{t^2 - \rho^2}} dt + \int_0^\rho \frac{\sqrt{a^2 - t^2}}{\sqrt{\rho^2 - t^2}} dt - \frac{\pi}{2} a\theta \operatorname{th} \pi\theta \right] \tag{4.3}$$

The displacements at the center of the circle are expressed by elementary functions

$$u = 0, \qquad w = 4\pi qHa\theta \frac{1 + \operatorname{ch} \pi\theta}{\operatorname{sh} 2\pi\theta} \tag{4.4}$$

The values of the stresses in the fixed part of the boundary of the half-space as well as the displacements inside the circle have been computed on the electronic computer

Table 1

ρ/a	$-\sigma/q \times 10^4$			$\tau/q \times 10^4$		
	steel	concrete	sandstone	steel	concrete	sandstone
1.01	34 856	33 797	34 440	12 689	17 988	16 638
1.10	6 752	6 859	6 796	1 364	1 966	1 801
1.20	3 414	3 494	3 446	548	791	723
1.30	2 139	2 196	2 162	294	425	388
1.50	1 083	1 116	1 096	119	172	157
1.80	521	538	528	45	65	60
2.00	354	366	359	27	39	36
4.00	36	38	37	1	2	2
10.00	2	2	2	0	0	0

Table 2

ρ/a	$w/4qHa \times 10^4$			$-u/4qH\alpha a \times 10^4$		
	steel	concrete	sandstone	steel	concrete	sandstone
0.00	9652	9337	9523	0	0	0
0.10	9602	9287	9473	545	539	542
0.20	9451	9137	9322	1085	1073	1080
0.30	9194	8881	9066	1614	1596	1607
0.40	8822	8512	8694	2123	2101	2115
0.50	8320	8013	8194	2606	2574	2593
0.60	7664	7362	7540	3046	3007	3030
0.70	6813	6517	6691	3416	3367	3396
0.80	5682	5398	5565	3656	3596	3632
0.90	4064	3803	3956	3588	3511	3557
0.95	2852	2616	2755	3216	3129	3181
0.99	1196	1025	1125	2106	2009	2067
1.00	0	0	0	0	0	0

"Odra-1204". In order to verify the cases of complete isotropy, the normal stresses have also been computed by the formulas obtained in [2]. Within the limits of the accepted relative accuracy of the computations, equal to 0.01, the results of the calculations by the formulas of this paper and by the formulas of [2] have turned out to be the same, therefore they are not given here. The values of the elastic constants for steel, concrete and sandstone have been taken as in [7]; the notation B, C, ν_1 and ν_2 used in [7] correspond here to $(2\pi H)^{-1}$, α, γ_1 and γ_2. In Table 1 we have given the results of the computations for the dimensionless stresses and in Table 2 for the dimensionless displacements. It should be noted, that in the adopted form the dimensionless displacements and also the normal stresses depend only weakly on the properties of the material of the half-space.

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